Symmetry properties under arbitrary field redefinitions of the metric energy–momentum tensor in classical field theories and gravity

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Abstract

We derive a generic identity which holds for the metric (i.e. variational) energy–momentum tensor under any field transformation in any generally covariant classical Lagrangian field theory. The identity determines the conditions under which a symmetry of the Lagrangian is also a symmetry of the energy–momentum tensor. It turns out that the stress tensor acquires the symmetry if the Lagrangian has the symmetry in a generic curved spacetime. In this sense a field theory in flat spacetime is not self–contained. When the identity is applied to the gauge invariant spin-two field in Minkowski space, we obtain an alternative and direct derivation of a known no–go theorem: a linear gauge invariant spin-2 field, which is dynamically equivalent to linearized General Relativity, cannot have a gauge invariant metric energy–momentum tensor. This implies that attempts to define the notion of gravitational energy density in terms of the metric energy–momentum tensor in a field–theoretical formulation of gravity must fail.

1 Introduction

Total energy and energy density are clearly among the most significant physical quantities in any field theory. The lesson we have learnt from Einstein's General Relativity

is that the adequate description of energy and momentum of any kind of matter and field, except for the gravitational field itself, is in terms of the variational (with respect to the spacetime metric) energy–momentum tensor(the metric stress tensor, for short).

In the gauge theories of particle physics the metric stress tensors for the gauge fields are all gauge invariant. This may arouse a conviction that this is a generic feature of any gauge invariant theory. However this is not the case. In general the metric stress tensor does not inherit the gauge–independence property of the underlying Lagrangian. The most important example is linearized General Relativity. This theory is dynamically equivalent to the linear massless spin–two field, whose metric stress tensor is gauge dependent [1]. Then two related problems arise. Why does the metric stress tensor for a gauge invariant field in Minkowski space lose this symmetry? Is it possible to construct a linear spin–two field theory in flat spacetime with a gauge invariant metric stress tensor?

An answer to the first problem follows from a "folk theorem", rigorously stated and proven by Deser and McCarthy in [2] to the effect that the Poincaré generators, being spatial integrals of the metric stress tensor, are gauge invariant and thus unique. The theorem shows that in quantum field theory, where only global (i.e. integrals over all 3–space) quantities, such as total energy and momentum of a quantum system, are physical (measurable) ones, the inevitable gauge dependence of the metric stress tensor (for fields carrying spin larger than one) is quite harmless. It follows from the proof that the gauge dependence of this tensor is due to the fact that the gauge transformations involve the spacetime metric. The negative answer to the second problem is also contained in that work: it is stated there that this gauge dependence is unavoidable (see also sect. 4 below).

In a classical gauge invariant field theory any gauge dependence of the metric stress tensor is truly harmful since this tensor cannot act as the source in Einstein field equations and this defect makes the theory unphysical. Even if such a field is viewed as a test one in a fixed spacetime, its theory remains defective since the local conserved currents (which exist if there are Killing vectors) do not determine physical flows of energy or momentum through a boundary of a spatially bounded region.

One may consider arbitrary field redefinitions depending on additional quantities and some of these may be symmetries of specific Lagrangians. It is interesting to see how generic is the case of gauge symmetry breaking by the metric stress tensor for fields with spins $s > 1$. To get an overall and unified picture we investigate in the present paper symmetry properties of the stress tensor in any classical generally covariant Lagrangian field theory. We establish conditions under which any symmetry (which is "internal" in the sense that it is not mere covariance under coordinate transformations) of the underlying Lagrangian becomes the symmetry of the metric stress tensor. To this end we derive a generic identity, valid for any Lagrangian in a curved spacetime for an arbitrary field transformation (being a symmetry or not), which determines how the stress tensor is altered under this transformation. This identity is the thrust of the paper. For a

symmetry transformation of the Lagrangian the identity shows that the metric stress tensor inherits the symetry property provided that either the field equations hold or the transformation is metric independent. Thus it turns out that the spacetime metric plays a key role for all symmetries in a field theory and not only for gauge invariance. Furthermore it follows from the identity that if a symmetry of the field Lagrangian holds only in flat spacetime (more generally: in a narrow class of curved spacetimes) and is broken in a generic curved one (actually in an open neighbourhood of flat spacetime or of that class, respectively), then the symmetry is broken for the metric stress tensor even in Minkowski space (this class of spacetimes).

As a first application of the identity we provide a different derivation of the result already given in [2] that all known metric stress tensors for a linear spin-two field (and for higher spins) in flat spacetime are gauge dependent. Next the identity furnishes an alternative (to that implicit in [2]), simple and direct proof of a no–go theorem: if in the weak–field approximation gravity is described by a gauge invariant linear and symmetric tensor field and its stress tensor is quadratic and contains no higher than second derivatives of the field, then this tensor cannot be gauge invariant. If the assumptions are valid, then no meaningful notion of local gravitational energy can arise in this way. Since the identity works equally well for any other field symmetry we expect that it will find further nontrivial applications.

The paper is organized as follows. Section 2 is the heart of the paper: we derive there the identity for the stress tensor variation under any field redefinition. In section 3 we study the gauge invariant linear massless spin-two field in Minkowski space and show, as a direct consequence of the identity, that the metric stress tensor breaks the gauge symmetry. Following the same way the no–go theorem for the weak–field approach to linear gravity is proven in section 4. Conclusions are contained in section 5.

2 Transformation of the metric stress tensor under a field redefinition

Let ϕ be a dynamical field or a multiplet of fields (indices suppressed) described by a generally covariant action functional with a Lagrangian density $\sqrt{-g}L(\phi, g_{\mu\nu})$, residing in a curved spacetime with a (dynamical or background) metric $g_{\mu\nu}$; for simplicity we assume that L does not depend on second and higher derivatives of ϕ. Let $\phi = \varphi(\phi', \xi, g)$ be any invertible transformation of the dynamical variable, which in general involves the metric tensor[1] and a non–dynamical vector or tensor field ξ and its first covariant derivative $\nabla\xi$. The transformation is arbitrary with the exception that we exclude the

[1]Throughout the paper the symbol g has two distinct meanings: it denotes either the determinant of the metric, or the metric itself. The ambiguity should not cause any confusion. The covariant derivatives $\nabla_\alpha f \equiv f_{;\alpha}$ are taken with respect to the spacetime metric $g_{\mu\nu}$.

tensor (or spinor) transformations of the field under a mere coordinate transformation. As a consequence, as opposed to many authors (see e.g. [3], [4], [5]), we do not view the transformation group of the dynamical variables induced by spacetime diffeomorphisms as a gauge group. The transformation need not be infinitesimal. Under the change of the dynamical field one sets

$$L(\phi, g) = L(\varphi(\phi', \xi, g), g) \equiv L'(\phi', \xi, g). \tag{1}$$

The variational (metric) stress tensor[2] (signature is $-+++$) is defined as usual by

$$T_{\mu\nu}(\phi, g) \equiv -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L)}{\delta g^{\mu\nu}}. \tag{2}$$

As is well known, the definition of variational derivative implies that the tensor $T_{\mu\nu}$ is not affected by adding a total divergence to the Lagrangian, a property that we shall use in the sequel. In concrete computations, one obtains the variational stress tensor by taking the variation of the action functional, then factorizing the variation $\delta g^{\mu\nu}$ of the metric: this requires dropping a total divergence to get rid of terms containing covariant derivatives of $\delta g^{\mu\nu}$. Having in mind this standard procedure, in the subsequent computations we rely on the following expression, equivalent to (2):

$$\delta_g(\sqrt{-g}L) = -\frac{1}{2}\sqrt{-g}\left[T_{\mu\nu}(\phi, g)\delta g^{\mu\nu} + \mathrm{div}\right], \tag{3}$$

where div means a full divergence which we will sometimes omit, as this term plays no role in our discussion; we will mark its presence from time to time to display an exact equality. In evaluating the variation in eq. (3) one assumes that ϕ is a fundamental field, i.e. is *not* affected by metric variations, $\delta_g\phi = 0$. This is the case of the vector potential (one–form) A_μ in electrodynamics while A^μ is already metric dependent with $\delta_g A^\mu = A_\nu \delta g^{\mu\nu}$. Hence, in evaluating $\delta_g L$ one takes into account only the explicit dependence of L on $g_{\mu\nu}$ and $g_{\mu\nu,\alpha}$ (or covariantly, on $g_{\mu\nu}$ and $\Gamma^{\alpha}_{\mu\nu}$).

In terms of the new field ϕ' and of the transformed Lagrangian L', the stress tensor of the theory is re-expressed as follows:

$$\delta_g(\sqrt{-g}L'(\phi', \xi, g)) \equiv -\frac{1}{2}\sqrt{-g}\left[T'_{\mu\nu}(\phi', \xi, g)\delta g^{\mu\nu} + \mathrm{div}\right]. \tag{4}$$

To evaluate $T'_{\mu\nu}(\phi', \xi, g)$ one assumes that the appropriate (covariant or contravariant) components of the field ξ are metric independent, i.e. $\delta_g\xi = 0$, while metric

[2] We emphasize that L is a matter Lagrangian. Whenever the metric is regarded as a dynamical field one should add to L a separate gravitational Lagrangian L_g. Our investigations are independent of the form of gravitational field equations and whether they hold at all, thus we do not take L_g into account.

variations of the new dynamical field ϕ' are determined by the inverse transformation $\phi' = \varphi^{-1}(\phi, \xi, g)$, i.e.

$$\delta_\varphi \phi' = \frac{\partial \varphi^{-1}}{\partial g^{\mu\nu}} \delta g^{\mu\nu} + \frac{\partial \varphi^{-1}}{\partial g^{\mu\nu}{}_{,\alpha}} \delta g^{\mu\nu}{}_{,\alpha}. \tag{5}$$

We denote this variation by $\delta_\varphi \phi'$ to emphasize that ϕ' and $g_{\mu\nu}$ are *not* independent fields: the value $\phi'(p)$ at any point p depends both on $\phi(p)$ and $g_{\mu\nu}(p)$. Any scalar or tensor function $f(\phi', \nabla\phi', g)$ depends on the metric both explicitly (including the connection Γ) and implicitly via ϕ', therefore its metric variation is determined by the *substantial (*or *total) variation* $\overline{\delta}_g$,

$$\overline{\delta}_g f \equiv \delta_g f + \delta_\varphi f. \tag{6}$$

Here $\delta_g f$ is the variation taking into account only the explicit metric dependence of f, i.e.

$$\delta_g f \equiv \frac{\partial f}{\partial g^{\mu\nu}} \delta g^{\mu\nu} + \frac{\partial f}{\partial \nabla\phi'} \delta_g \nabla\phi'. \tag{7}$$

To compute $\delta_g \nabla\phi'$ one writes symbolically $\nabla\phi' = \partial\phi' - \phi'\Gamma$ (assuming that ϕ is a covariant tensor) and recalling that δ_g does not act on ϕ' one gets

$$\delta_g \nabla\phi' = -\phi' \delta\Gamma. \tag{8}$$

On the other hand the variation δ_φ takes into account the metric dependence of f via $\phi' = \varphi^{-1}(\phi, \xi, g)$, then

$$\delta_\varphi f \equiv \frac{\partial f}{\partial \phi'} \delta_\varphi \phi' + \frac{\partial f}{\partial \nabla\phi'} \delta_\varphi \nabla\phi' \tag{9}$$

with $\delta_\varphi \phi'$ given by (5) and

$$\begin{aligned} \delta_\varphi \nabla\phi' &= \delta_\varphi(\partial\phi' - \Gamma\phi') = \partial\delta_\varphi\phi' - \Gamma\delta_\varphi\phi' \\ &= \nabla\delta_\varphi\phi', \end{aligned} \tag{10}$$

thus δ_φ commutes with the covariant derivative ∇. For a function $f(\phi, \nabla\phi, g)$ the operators $\overline{\delta}_g$ and δ_g coincide, i.e.

$$\begin{aligned} \overline{\delta}_g f(\phi, \nabla\phi, g) &= \delta_g f(\phi, \nabla\phi, g) \\ &= \frac{\partial f}{\partial g^{\mu\nu}} \delta g^{\mu\nu} + \frac{\partial f}{\partial \nabla\phi} \delta_g \nabla\phi, \end{aligned} \tag{11}$$

where

$$\delta_g \nabla\phi = \delta_g(\partial\phi - \Gamma\phi) = -\phi\delta\Gamma. \tag{12}$$

Accordingly, δ_g on the l.h.s. of eq. (4) should be replaced by $\overline{\delta}_g$.

The identity (1), which is valid for *all* ϕ, $g_{\mu\nu}$ and transformations φ, implies

$$\delta_g L(\phi, g) = \overline{\delta}_g L'(\phi', \xi, g), \tag{13}$$

what in turn implies the crucial equality

$$T_{\mu\nu}(\phi, g) = T'_{\mu\nu}(\phi', \xi, g); \tag{14}$$

in general the two tensors depend differently on their arguments. We stress that in this way we have *not* constructed the stress tensor for the *different* theory (also described by the Lagrangian L') in which ϕ' would represent a new, metric–independent field variable; this tensor would *not* coincide with $T_{\mu\nu}$ unless the fields ϕ and ϕ', related by the transformation φ, are also *solutions* of the respective field equations: an expression for this new stress tensor can be found, for instance, in [6] (notice that their equation (2.11) holds whenever the transformation does not depend on the derivatives of the metric, otherwise an additional term occurs). Here, we are dealing with field redefinitions *within* the same theory, not with transformations relating theories which are dynamically equivalent but different in their physical interpretation[3].

We are interested first in finding out a generic relationship between $T_{\mu\nu}(\phi, g)$ and $T_{\mu\nu}(\phi', g)$ and then in its reduced version in the case of a metric–dependent symmetry transformation $\phi = \varphi(\phi', \xi, g)$ with a specific φ. To this end we explicitly evaluate $T'_{\mu\nu}(\phi', \xi, g)$ from the definition (4) and then apply the equality (14).
It is convenient to write the transformed Lagrangian as a sum

$$L'(\phi', \xi, g) \equiv L(\phi', g) + \Delta L'(\phi', \xi, g), \tag{15}$$

this is a definition of $\Delta L'(\phi', \xi, g)$. This splitting allows one to obtain $T_{\mu\nu}(\phi', g)$ upon applying $\overline{\delta}_g$ to eq. (15). It is furthermore convenient to make the inverse transformation in the term $\Delta L'$, then

$$\Delta L'(\phi', \xi, g) = \Delta L'(\varphi^{-1}(\phi, \xi, g), \xi, g) \equiv \Delta L(\phi, \xi, g), \tag{16}$$

and this is a definition of $\Delta L(\phi, \xi, g)$. Then eq. (15) takes the form

$$L'(\phi', \xi, g) = L(\phi', g) + \Delta L(\phi, \xi, g). \tag{17}$$

The Lagrange equations for ϕ arising from $L(\phi, g)$ are

$$E(\phi) \equiv \frac{\delta L}{\delta \phi} = \frac{\partial L}{\partial \phi} - \nabla \left(\frac{\partial L}{\partial \nabla \phi} \right) = 0; \tag{18}$$

[3] As far as one takes the variation of L' with respect to ϕ' to derive the field equations, it is irrelevant to state whether this field is metric independent or not: the resulting equations are the same, and this may be the reason why the distinction between the two viewpoints is usually not remarked in the literature. However, the variational derivatives of L' w. r. to $g^{\mu\nu}$ are different in the two cases.

the tensor $E(\phi)$ has the same rank and symmetry as the field ϕ, and the definition (3) provides the energy–momentum tensor for ϕ,

$$T_{\mu\nu}(\phi, g)\delta g^{\mu\nu} = \delta g^{\mu\nu}(g_{\mu\nu}L - 2\frac{\partial L}{\partial g^{\mu\nu}}) - 2\frac{\partial L}{\partial \nabla\phi}\delta_g \nabla\phi. \tag{19}$$

We can now evaluate $T'_{\mu\nu}(\phi', \xi, g)$:

$$\begin{aligned}\overline{\delta}_g(\sqrt{-g}L'(\phi', \xi, g)) \quad &= \quad -\frac{1}{2}\sqrt{-g}[g_{\mu\nu}\delta g^{\mu\nu}(L(\phi', g) + \Delta L(\phi, \xi, g)) \\ &\quad -2\overline{\delta}_g(L(\phi', g) + \Delta L)].\end{aligned}$$

Here from eqs. (6), (7) and (9)

$$\overline{\delta}_g L(\phi', g) = \left[\frac{\partial L(\phi')}{\partial g^{\mu\nu}}\delta g^{\mu\nu} + \frac{\partial L(\phi')}{\partial \nabla\phi'}\delta_g \nabla\phi'\right] + \left[\frac{\partial L(\phi')}{\partial \phi'}\delta_\varphi \phi' + \frac{\partial L(\phi')}{\partial \nabla\phi'}\delta_\varphi \nabla\phi'\right],$$

and the first square bracket contributes to $T_{\mu\nu}(\phi', g)$ while the second one is equal, after applying (10), to

$$\left[\frac{\partial L(\phi')}{\partial \phi'}\delta_\varphi \phi' - \delta_\varphi \phi' \nabla\left(\frac{\partial L(\phi')}{\partial \nabla\phi'}\right) + \mathrm{div}\right]. \tag{20}$$

Employing (11) one has $\overline{\delta}_g \Delta L(\phi, \xi, g) = \delta_g \Delta L$. Then employing eqs. (19), (18) and (14) and dropping a full divergence one arrives at the fundamental relationship

$$\begin{aligned}&\delta g^{\mu\nu}[T_{\mu\nu}(\phi', g) - T_{\mu\nu}(\phi, g) + g_{\mu\nu}\Delta L(\phi, \xi, g)] \\ &\qquad -2E(\phi')\delta_\varphi \phi' - 2\delta_g \Delta L(\phi, \xi, g) \quad = \quad 0.\end{aligned} \tag{21}$$

This is an identity (up to a total divergence) valid for any field, Lagrangian and any field transformation. The two terms containing ΔL can be written together as

$$-\frac{2}{\sqrt{-g}}\delta_g(\sqrt{-g}\Delta L). \tag{22}$$

We remark that if all full divergence terms were kept in the derivation of the identity, a divergence term would replace zero on the r.h.s. of eq. (21). However a total divergence cannot cancel the last three terms on the l.h.s. of the identity since in general these terms do not sum up into a divergence. Therefore the difference $T_{\mu\nu}(\phi', g) - T_{\mu\nu}(\phi, g)$ does not vanish in general.

The transformation $\phi \mapsto \phi'$ is a *symmetry transformation* of the theory (of the Lagrangian) iff $L'(\phi', \xi, g) = L(\phi', g) + \mathrm{div}$, i.e. if $\Delta L(\phi, \xi, g) = \mathrm{div}$ or is zero. According to the proposition "the metric variation of a divergence is another divergence", adding a covariant divergence to $L(\phi, g)$ does not affect the variational stress tensor; in a similar

way one shows that the variation with respect to the dynamical field ϕ of a full divergence gives rise to another divergence, thus the Lagrange field equations remain unaffected too. The equality $L'(\phi', \xi, g) = L(\phi', g) + \mathrm{div}$ should hold identically for a symmetry independently of whether the field equations are satisfied or not. It is worth stressing that we impose no restrictions on the transformation φ and on possible symmetries — they should only continuously (actually smoothly) depend on a set of parameters corresponding to a set of scalar functions or to the components of a vector or tensor field ξ; discrete transformations, like reflections, are excluded. The identity (21) has deeper consequences usually when the transformation φ depends on the spacetime metric (possibly through covariant derivatives of the field ξ). For any symmetry the identity (21) reduces to

$$\delta g^{\mu\nu}[T_{\mu\nu}(\phi', g) - T_{\mu\nu}(\phi, g)] - 2E(\phi')\delta_\varphi \phi' = 0 \tag{23}$$

since for $\Delta L = \mathrm{div} \equiv \nabla_\alpha A^\alpha(\phi, \xi, g)$ one has

$$-\frac{2}{\sqrt{-g}}\delta_g(\sqrt{-g}\Delta L) = -2\nabla_\alpha(\delta_g A^\alpha - \frac{1}{2}A^\alpha g_{\mu\nu}\delta g^{\mu\nu}) \tag{24}$$

and the divergence is discarded. As a trivial example, consider Maxwell electrodynamics. Here $\phi = A_\mu$, $\phi' = A_\mu + \partial_\mu f$ with arbitrary f; since the gauge transformation is metric independent, $\delta_\varphi \phi' = 0$. Then the term $E(\phi')\delta_\varphi \phi' = \nabla_\nu F^{\mu\nu}\delta_\varphi \phi'$ vanishes giving rise to the gauge invariance of $T_{\mu\nu}$ independently of Maxwell equations. A nontrivial example is considered in the next section.

Since the term $E(\phi')\delta_\varphi \phi'$ is different from zero in general (i.e. for fields not being solutions and for metric–dependent symmetry transformations), one arrives at the conclusion: *the metric energy–momentum tensor for a theory having a symmetry does not possess this symmetry*. The symmetry implies that

$$L(\phi, g) = L(\phi', g) + \mathrm{div}, \tag{25}$$

i.e. replacing ϕ by ϕ' in the Lagrangian alters the action

$$S[\phi, g] = \int d^4x\sqrt{-g}L(\phi, g) \tag{26}$$

by at most a surface term. Yet the associated integral

$$\delta_g S = -\frac{1}{2}\int d^4x\sqrt{-g}T_{\mu\nu}(\phi, g)\delta g^{\mu\nu} \tag{27}$$

does depend on the transformation. It is only *for solutions*, $E(\phi') = E(\phi) = 0$, that the energy–momentum tensor does possess the same symmetry, $T_{\mu\nu}(\phi', g) = T_{\mu\nu}(\phi, g)$.
In physics one is mainly interested in quantities built up of solutions of equations of motion, but from the mathematical viewpoint it is worth noticing that the symmetry property is *not* carried over from S to $\delta_g S$.

3 Linear massless spin–2 field in Minkowski space

In gauge theories of particle physics the field potentials are exterior forms since the fields carry spin one. Then the gauge transformations are independent of the spacetime metric and the identity (23) implies gauge invariance of the stress tensor for arbitrary fields, not necessarily being solutions to the field equations. Yet it is characteristic for gauge theories that *for integer spins larger than one* a gauge transformation necessarily involves covariant derivatives of vector or tensor fields [7], giving rise to gauge dependent energy–momentum tensors. However the gauge symmetry in these theories holds only in flat spacetime (more precisely, the Lagrangians are gauge invariant only in Minkowski space, while Lagrange field equations are gauge invariant in empty spacetimes, i.e. for $R_{\mu\nu} = 0$) and disappears in a generic curved one. Yet the stress tensors are gauge dependent for solutions even in Minkowski space. This case is comprised in the generic theory developed in the previous section, identity (21), but is rather misleading and confusing and because of its relevance for gravity it deserves a separate treatment.

We illustrate the effect in the case of spin–two field $\psi_{\mu\nu} = \psi_{\nu\mu}$. Any Lagrangian $L(\psi_{\mu\nu}, \psi_{\mu\nu;\alpha}, g_{\alpha\beta})$ generates equations of motion of the form

$$E^{\mu\nu}(\psi) \equiv \frac{\delta L}{\delta\psi_{\mu\nu}} = \frac{\partial L}{\partial\psi_{\mu\nu}} - \nabla_\alpha(\frac{\partial L}{\partial\psi_{\mu\nu;\alpha}}) = 0 \tag{28}$$

and the expression (19) for the stress tensor reads now

$$\begin{aligned} T_{\mu\nu}(\psi, g) &= g_{\mu\nu}L - 2\frac{\partial L}{\partial g^{\mu\nu}} + \\ &\qquad 2\nabla_\sigma[2\frac{\partial L}{\partial\psi_{\beta(\alpha;\sigma)}}\psi_{\beta(\mu}g_{\nu)\alpha} - \frac{\partial L}{\partial\psi_{\beta(\alpha;\tau)}}\psi^\sigma{}_\beta g_{\alpha(\mu}g_{\nu)\tau}]. \end{aligned} \tag{29}$$

Let L be gauge–invariant under $\psi_{\mu\nu} \mapsto \psi'_{\mu\nu} \equiv \psi_{\mu\nu} + \xi_{\mu;\nu} + \xi_{\nu;\mu}$ in *flat spacetime*. Using covariant expressions in Minkowski space (coordinates need not be Cartesian ones) this symmetry implies that

$$L(\psi, g) = L(\psi', g) + \nabla_\alpha A^\alpha(\psi, \xi, g) \tag{30}$$

for some vector A^α *provided that* the covariant derivatives commute, $\nabla_\mu\nabla_\nu = \nabla_\nu\nabla_\mu$. In a generic curved spacetime the "gauge" transformation is no more a symmetry since

$$L(\psi, g) = L(\psi', g) + \mathrm{div} + \Delta L(\psi, \xi, g), \tag{31}$$

with div denoting a full divergence while ΔL is a sum of terms proportional to Riemann tensor. For $\delta_g\psi_{\mu\nu} = 0 = \delta_g\xi_\mu$ one gets for this gauge transformation

$$\delta_\varphi\psi'_{\mu\nu} = -2\xi_\alpha\delta\Gamma^\alpha_{\mu\nu}. \tag{32}$$

In this case the fourth term in the identity (21) reads

$$-2E^{\mu\nu}(\psi')\delta_{\varphi}\psi'_{\mu\nu} = 4E^{\mu\nu}(\psi')\xi_{\alpha}\delta\Gamma^{\alpha}_{\mu\nu} \tag{33}$$

and discarding the full divergence arising from $\delta g^{\mu\nu}{}_{;\alpha}$ one arrives at the following explicit form of (21) for the linear massless spin–2 field $\psi_{\mu\nu}$,

$$\begin{aligned}\delta g^{\mu\nu}\{T_{\mu\nu}(\psi', g) - T_{\mu\nu}(\psi, g) + g_{\mu\nu}\Delta L(\psi, \xi, g)\\ +2\nabla_{\alpha}[2\xi_{(\mu}E^{\alpha}_{\nu)}(\psi', g) - \xi^{\alpha}E_{\mu\nu}(\psi', g)]\}\\ -2\delta_g\Delta L(\psi, \xi, g) &= 0.\end{aligned} \tag{34}$$

We recall that this identity is valid only for the transformation $\psi'_{\mu\nu} = \psi_{\mu\nu} + 2\xi_{(\mu;\nu)}$. One is interested in evaluating this identity in flat spacetime, $R_{\alpha\beta\mu\nu}(g) = 0$, what will be symbolically denoted by $g = \eta$. One has $\Delta L|_{g=\eta} = \mathrm{div}$ while $\delta_g\Delta L|_{g=\eta} \neq 0$. In fact, ΔL is a sum of terms of the form $t^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}$ where $t^{\alpha\beta\mu\nu}$ is made up of $\psi_{\mu\nu}$, $\psi_{\mu\nu;\alpha}$, ξ_{α}, $\xi_{\alpha;\mu}$ and $\xi_{\alpha;\mu\nu}$ (see eq. (42) below). Then $\delta_g\left(t^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}\right)\Big|_{g=\eta} = 0 + t^{\alpha\beta\mu\nu}\delta R_{\alpha\beta\mu\nu}\Big|_{g=\eta}$ and the latter term does not vanish, in general, even for $R_{\alpha\beta\mu\nu}(g) = 0$.

Let us denote the expression in square backets in (34) by $F^{\alpha}_{\mu\nu}(\psi', \xi, g)$. In flat spacetime the gauge invariance implies (for any $\psi_{\mu\nu}$, not necessarily a solution) that $E_{\mu\nu}(\psi', \eta) = E_{\mu\nu}(\psi, \eta)$ and then $\nabla_{\alpha}F^{\alpha}_{\mu\nu}(\psi', \xi, g)\Big|_{g=\eta} = \nabla_{\alpha}F^{\alpha}_{\mu\nu}(\psi, \xi, \eta)$. Assuming that $\psi_{\mu\nu}$ is a solution in Minkowski space, $E_{\mu\nu}(\psi, \eta) = 0$, one gets that $\nabla_{\alpha}F^{\alpha}_{\mu\nu}(\psi, \xi, \eta) = 0$. Thus, the indentity (34) reduces for solutions of the fields equations, in Minkowski space, to

$$\delta g^{\mu\nu}\{T_{\mu\nu}(\psi', \eta) - T_{\mu\nu}(\psi, \eta)\} - 2\,\delta_g\Delta L(\psi, \xi, g)|_{g=\eta} = 0. \tag{35}$$

This relationship (not an identity) shows that the stress tensor is *not* gauge invariant even in flat spacetime. In other terms, the symmetry properties of the Lagrangian in Minkowski space are insufficient for determining symmetry properties of the metric stress tensor in this spacetime. To this end one must investigate the Lagrangian in a general curved spacetime and its behaviour there is relevant for the stress tensor in flat spacetime.

The gauge invariant linear massless spin–2 field was introduced by Fierz and Pauli [8] and its Lagrangian usually bears their names. Finding out the appropriate Lagrangian is not straightforward and they had to use a rather indirect procedure. For our purposes it is adequate to employ the dynamical equivalence of this field to linearized General Relativity and following Aragone and Deser [9] generate a Lagrangian for it as the Lagrangian for a metric perturbation around Minkowski space. The resulting Lagrangian is of first order and may be unambiguously expressed in any curved background as

$$L_W(\psi, g) = \frac{1}{4}\left(-\psi^{\mu\nu;\alpha}\psi_{\mu\nu;\alpha} + 2\psi^{\mu\nu;\alpha}\psi_{\alpha\mu;\nu} - 2\psi^{\mu\nu}{}_{;\nu}\psi_{;\mu} + \psi^{;\mu}\psi_{;\mu}\right) \tag{36}$$

with $\psi = g^{\mu\nu}\psi_{\mu\nu}$. This Lagrangian appeared first in the textbook [10] and will be referred hereafter to as *Wentzel Lagrangian*. Actually in Minkowski space the choice of a Lagrangian for $\psi_{\mu\nu}$ is not unique and a number of equivalent Lagrangians exist[4]. For example one can replace the second term in (36) by a more symmetric term $\psi^{\mu\nu}{}_{;\nu}\psi_{\mu\alpha}{}^{;\alpha}$ and the resulting Lagrangian L_S differs from L_W by a full divergence. However in a curved spacetime the two Lagrangians differ by a curvature term, $L_S(\psi, g) = L_W + \text{div} + H$, where $H \equiv \psi^{\nu}{}_{\mu}\psi^{\mu\alpha}{}_{;[\nu\alpha]} = \frac{1}{2}\psi^{\alpha\beta}(\psi_{\beta}{}^{\mu}R_{\mu\alpha} + \psi^{\mu\nu}R_{\mu\alpha\beta\nu})$. There exists also a second–order Lagrangian

$$L_{II}(\psi, g) = -\frac{1}{2}\psi^{\mu\nu}G^{\mathrm{L}}_{\mu\nu}(\psi, g) \tag{37}$$

where

$$\begin{aligned} G^{\mathrm{L}}_{\mu\nu}(\psi, g) \;\; &\equiv \;\; \frac{1}{4}\big(-\psi_{\mu\nu;\alpha}{}^{;\alpha} + \psi^{\alpha}{}_{\mu;\nu\alpha} + \psi^{\alpha}{}_{\nu;\mu\alpha} - \psi_{;\mu\nu} \\ &\quad -g_{\mu\nu}\psi_{\alpha\beta}{}^{;\alpha\beta} + g_{\mu\nu}\psi_{;\alpha}{}^{;\alpha}\big) \end{aligned} \tag{38}$$

is the Einstein tensor linearized around Minkowski space and then formally written down for an arbitrary background (actually, if $G_{\mu\nu}$ is linearized around a curved background, then there appear in the expansion additional terms depending on the background curvature). This Lagrangian is equivalent (also in a curved spacetime) to Wentzel Lagrangian, $L_{II}(\psi, g) = L_W(\psi, g) + \text{div}$.

The Lagrange equations of motion arising from L_W (or equivalently from L_{II}) in any spacetime are

$$E_{\mu\nu}(\psi, g) = -G^{\mathrm{L}}_{\mu\nu}(\psi, g) = 0, \tag{39}$$

while those resulting from L_S contain additional curvature terms. However, as is well known ([1],[9]), the linear spin–2 field is inconsistent in a curved spacetime since both sets of field equations develop additional constraints due to the curvature. We shall assume that eqs. (39) hold only in flat spacetime.

The inequivalence of L_W and L_S (and most other Lagrangians) raises the problem of which Lagrangian should be used, since the stress tensors associated with these Lagrangians will be different even in flat spacetime. The inconsistency of the theory in the presence of a gravitational field shows that a proper choice does not exist. Equivalence of L_W and L_{II} points rather to L_W and we shall employ Wentzel Lagrangian. Actually the problem of gauge dependence of any stress tensor is independent of the choice; in each case computations are of the same length and give the same outcome.

The metric energy–momentum tensor generated by Wentzel Lagrangian is

$$T^{W}_{\mu\nu}(\psi, \eta) \;\; = \;\; 2\psi_{\alpha(\mu}\psi_{\nu)\beta}{}^{;\alpha\beta} - \psi_{\mu\nu;\alpha\beta}\psi^{\alpha\beta} - \psi_{\alpha(\mu}\psi_{;\nu)}{}^{;\alpha} + \frac{1}{2}\psi_{\mu\nu}\Box\psi$$

[4] In fact the requirement of gauge invariance uniquely fixes a quadratic Lagrangian for the field up to some boundary terms [11].

$$
-\psi_{\mu\nu}\psi^{\alpha\beta}{}_{;\alpha\beta}+\frac{1}{2}g_{\mu\nu}\psi^{\alpha\beta}\psi_{;\alpha\beta}-2\psi_{\alpha\beta;(\mu}\psi_{\nu)}{}^{\alpha;\beta}+\frac{1}{2}\psi_{\alpha\beta;\mu}\psi^{\alpha\beta}{}_{;\nu}
$$

$$
+2\psi_{(\mu}{}^{\alpha;\beta}\psi_{\nu)(\alpha;\beta)}-\psi_{\mu\nu;\alpha}\psi^{\alpha\beta}{}_{;\beta}-\frac{1}{2}\psi_{\mu\nu;\alpha}\psi^{;\alpha}+\psi_{;(\mu}\psi_{\nu)\alpha}{}^{;\alpha}
$$

$$
-\frac{1}{2}\psi_{;\mu}\psi_{;\nu}+\frac{1}{4}g_{\mu\nu}\left(-\psi^{\alpha\beta;\sigma}\psi_{\alpha\beta;\sigma}+2\psi^{\alpha\beta;\sigma}\psi_{\sigma\alpha;\beta}+\psi^{;\alpha}\psi_{;\alpha}\right), \tag{40}
$$

where $\Box\psi \equiv \nabla^\alpha\nabla_\alpha\psi$. This covariant expression holds only in flat spacetime since in deriving it one assumes that the covariant derivatives commute.

Now one can prove the gauge dependence of $T^W_{\mu\nu}(\psi,\eta)$ either directly from its explicit form (40) or employing the relationship (35). The latter method is more convenient if one wishes to show (as is done in the next section) that this deficiency is not peculiar to $T^W_{\mu\nu}$ but is a generic feature of all Lagrangians which are equivalent to L_W in flat spacetime.

Wentzel Lagrangian and any other Lagrangian equivalent to it is gauge invariant only in flat spacetime. In fact, writing (17) as

$$
L_W(\psi,g)=L_W(\psi',g)+\Delta L_W(\psi,\xi,g). \tag{41}
$$

one finds (disregarding a divergence) that

$$
\begin{aligned}
\Delta L_W(\psi,\xi,g) \;\; = \;\; & R_{\alpha\beta\mu\nu}(h^{\mu\beta;\alpha}\xi^\nu-h^{\nu\beta}\xi^{\mu;\alpha}) \\
& +R_{\alpha\beta}(h^{\alpha\mu}\xi^\beta{}_{;\mu}-h^{\alpha\mu}{}_{;\mu}\xi^\beta+2\xi^\alpha{}_{;\mu}\xi^{(\beta;\mu)}-\xi^{\alpha;\beta}\xi^\mu{}_{;\mu}) \\
& +R_{\mu\nu}(\xi_\alpha{}^{;\alpha\mu}\xi^\nu-\xi^{\mu;\nu\alpha}\xi_\alpha),
\end{aligned} \tag{42}
$$

where $h_{\mu\nu}\equiv\psi_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\psi$.

One notes in passing that the tensor $E_{\mu\nu}$ is gauge invariant in any empty spacetime ($R_{\mu\nu}=0$) [1],

$$
\begin{aligned}
E_{\mu\nu}(\psi',g) \;\; = \;\; & E_{\mu\nu}(\psi,g)-\xi^\alpha R_{\mu\nu;\alpha}-2\xi_{\alpha;(\mu}R_{\nu)}{}^\alpha+ \\
& +g_{\mu\nu}\left(\frac{1}{2}\xi^\alpha R_{;\alpha}+\xi^{\alpha;\beta}R_{\alpha\beta}\right),
\end{aligned} \tag{43}
$$

i.e. even in the backgrounds where the gauge symmetry of L_W is broken.

The variation $\delta_g\Delta L_W$ evaluated in Minkowski space is different from zero, showing the gauge dependence of $T^W_{\mu\nu}$. The expression is extremely involved; in the special case of the solution $\psi_{\mu\nu}=0$ it is shown in eq. (49) in the next section.

4 Nonexistence of a gauge invariant metric stress tensor and the problem of gravitational energy density

The fact that the stress tensor $T^W_{\mu\nu}$ depends on the gauge was known long ago [1] (actually Aragone and Deser investigated a Lagrangian different from L_W, giving rise to

a stress tensor different from $T^W_{\mu\nu}$, nevertheless the conclusion was clearly the same). More interesting is the problem whether there exists a Lagrangian $L_K(\psi, g)$, which is equivalent to L_W in Minkowski space, but which generates a different, gauge–invariant stress tensor $T^K_{\mu\nu}$ in this spacetime. The no–go theorem stating that such gauge invariant stress tensor does not exist was given in [2]. The authors of that work did not publish a detailed proof and only referred to the underlying "folk" wisdom. According to S. Deser, for all gauge fields (with metric dependent gauge transformations) in flat spacetime the manifest covariance of energy–momentum density objects is incompatible with their gauge invariance, i.e. these objects are either covariant or gauge invariant but not both [12]. In fact, one can always remove (in a non–covariant way) the non–physical components of the fields, so that the result will have no residual gauge dependence; notice however that this is not the same as producing a gauge–independent definition in the usual sense. Here we give an alternative direct proof of the no–go theorem based on the relationship (35).

We set by definition

$$L_K(\psi, g) = L_W(\psi, g) + K(\psi, g); \tag{44}$$

the term K should reduce to a full divergence in flat spacetime, while in a curved one it should contain a sum of terms proportional to the Riemann tensor:

$$\begin{aligned} K(\psi, g) &= \nabla_\mu A^\mu + a_1 R_{\alpha\beta\mu\nu}\psi^{\alpha\mu}\psi^{\beta\nu} + a_2 R_{\alpha\beta}\psi^{\alpha\beta}\psi + a_3 R_{\alpha\beta}\psi^{\alpha\mu}\psi^{\beta}{}_{\mu} \\ &\quad + a_4 R\psi^{\alpha\beta}\psi_{\alpha\beta} + a_5 R\psi^2. \end{aligned} \tag{45}$$

The curvature terms in K generate a nonvanishing contribution to $T^K_{\mu\nu}(\psi, \eta)$, possibly making it gauge independent for an appropriate choice of the constant coefficients $a_1, \ldots, a_5$. On the other hand, L_K and L_W give rise to the same Lagrange field equations in Minkowski space, $E_{\mu\nu}(\psi, \eta) = -G^{\mathrm{L}}_{\mu\nu}(\psi, \eta) = 0$.

The expression (45) is the most general one providing a physically acceptable stress tensor. In fact, we assume that

1. $T^K_{\mu\nu}$ should be exactly quadratic in $\psi_{\mu\nu}$. In fact, the term $K(\psi, g)$ is expected to cancel out the quadratic gauge–dependent terms in the stress tensor $T^W_{\mu\nu}$: if $K(\psi, g)$ contains cubic or linear terms, they would produce new gauge–dependent terms in the stress tensor which would not be compensated by terms arising from L_W.

2. $T^K_{\mu\nu}$ should contain at most second derivatives of $\psi_{\mu\nu}$. This is a physical postulate: the order of field derivatives in the energy–momentum tensor should not exceed the order of the Lagrange equations of motion.

The latter assumption implies that K should be linear in the Riemann tensor, and should not contain derivatives of the Riemann tensor itself; moreover, it should not

contain derivatives of the field $\psi_{\mu\nu}$ multiplied by the curvature components. Otherwise, it is easy to see that the metric variation of K would necessarily produce at least third derivatives of $\psi_{\mu\nu}$. Therefore, one should set $K(\psi, g) = \nabla_\mu A^\mu + k^{\alpha\beta\mu\nu} R_{\alpha\beta\mu\nu}$, the coefficients $k^{\alpha\beta\mu\nu}$ being functions of the field $\psi_{\mu\nu}$ not depending on its derivatives. The first assumption entails that these functions should be exactly quadratic in $\psi_{\mu\nu}$. Hence, we conclude that K cannot contain other terms besides those occurring in (45).

We now apply the formula (35) to $L_K(\psi, g)$: for solutions of the field equations in Minkowski space, one has

$$\delta g^{\mu\nu}\{T^K_{\mu\nu}(\psi', \eta) - T^K_{\mu\nu}(\psi, \eta)\} - 2\ \delta_g \Delta L_K(\psi, \xi, g)\big|_{g=\eta} = 0. \tag{46}$$

One concludes that the stress tensor $T^K_{\mu\nu}(\psi, \eta)$ is gauge invariant for solutions if and only if

$$\delta_g \Delta L_K\big|_{g=\eta} = \delta_g(\Delta L_W + \Delta K)\big|_{g=\eta} = 0. \tag{47}$$

Our aim is to show that eq. (47) cannot hold for arbitrary ξ_α. Under the gauge transformation K varies by

$$\begin{aligned}\Delta K(\psi, \xi, g) \;=\; & -a_1 R_{\alpha\beta\mu\nu}(2\psi^{\alpha\mu}\xi^{\beta\nu} + \xi^{\alpha\mu}\xi^{\beta\nu}) \\ & -R_{\alpha\beta}[a_2(\psi^{\alpha\beta}\xi + \psi\xi^{\alpha\beta} + \xi^{\alpha\beta}\xi) + a_3(2\psi^{\mu(\alpha}\xi^{\beta)}{}_\mu + \xi^{\alpha\mu}\xi^\beta{}_\mu)] \\ & -R[a_4(2\psi^{\alpha\beta}\xi_{\alpha\beta} + \xi^{\alpha\beta}\xi_{\alpha\beta}) + a_5(2\psi\xi + \xi^2)],\end{aligned} \tag{48}$$

where we introduced the abbreviations $\xi_{\mu\nu} \equiv \xi_{\mu;\nu} + \xi_{\nu;\mu}$ and $\xi \equiv g^{\mu\nu}\xi_{\mu\nu} = 2\xi^\mu{}_{;\mu}$.

If a gauge invariant $T^K_{\mu\nu}$ exists, eq. (47) should become an identity with respect to ξ_α for any solution $\psi_{\mu\nu}$ of $E_{\mu\nu}(\psi, \eta) = 0$. We show that this is not the case for the solution $\psi_{\mu\nu} = 0$. Even in this simplest case the expression for $\delta_g \Delta K$ in flat spacetime is too long to be presented here. The expression for $\delta_g \Delta L_W$ is shorter a bit and is worth exhibiting,

$$\begin{aligned}\delta_g \Delta L_W(0, \xi, g)\Big|_{g=\eta} \;=\; & \delta g^{\mu\nu}[\xi_{(\mu}{}^{;\alpha}\Box\xi_{\nu);\alpha} + \xi^\alpha{}_{;(\mu}\Box\xi_{\nu);\alpha} - 2\xi_{(\mu;\nu)\alpha\beta}\xi^{\alpha;\beta} \\ & -\xi_{(\mu;\nu)}\Box\xi^\alpha{}_{;\alpha} + g_{\mu\nu}\xi^{\alpha;\beta}\xi^\sigma{}_{;\sigma\alpha\beta} + \xi_{\mu;\alpha\beta}\xi_\nu{}^{;\alpha\beta} \\ & -\xi_{(\mu;\nu)\alpha}\Box\xi^\alpha - 2\xi_{(\mu;\nu)\alpha}\xi^{\beta;\alpha}{}_{;\beta} + \xi^\alpha{}_{;\alpha(\mu}\Box\xi_{\nu)} \\ & +\frac{1}{2}g_{\mu\nu}\xi^\sigma{}_{;\sigma\alpha}(\Box\xi^\alpha + \xi^{\beta;\alpha}{}_{;\beta})].\end{aligned} \tag{49}$$

Each term in eq. (49) should be separately cancelled by an appropriate term arising from $\delta_g \Delta K$, what implies that eq. (47) decomposes into 27 linear algebraic equations for the coefficients a_i. Among these, there are 10 linearly independent equations, and they clearly form and overdetermined system which admits no solution. This proves that eq. (47) cannot hold identically for an arbitrary vector field ξ_α and the stress tensor $T^K_{\mu\nu}$ *cannot* be gauge invariant, for any choice of the coefficients $a_1, \ldots, a_5$.

The same outcome can indeed be obtained by just evaluating the difference $T^K_{\mu\nu}(\psi', \eta) - T^K_{\mu\nu}(\psi, \eta)$ for K given in (45). In this case, however, the computation is much longer.

The physical relevance of the linear massless spin–two field $\psi_{\mu\nu}$ in flat spacetime stems from the fact that it is dynamically equivalent to linearized General Relativity. Hence this field is closely related to the problem of gravitational energy density. A conventional wisdom says that gravitational energy and momentum densities are non-measurable quantities. Nevertheless since the very advent of General Relativity there have been numerous attempts to construct a local concept of gravitational energy. A gravitational energy–momentum tensor is highly desirable for a number of reasons. For instance, it is emphasized in [13] that such a genuinely local tensor is required for a detailed description of cosmological perturbations in the early universe. Approaches to the problem in the framework of General Relativity include the use of a quasilocal energy–momentum tensor for a spatially bounded region (see e.g. [14]), various effective energy–momentum tensors which are conserved and may be invariant with respect to diffeomorphism–induced gauge transformations[5] [3], [4] and applications of the Noether approach with some version of the Belinfante symmetrization method [18]. Our considerations here do not apply to these objects.

A completely different approach to the problem is provided by the field theory approach to gravitation, according to which gravity is just a tensor field existing in Minkowski space, which is the spacetime of the physical world (for a historical review see [19]). In these theories of gravity the metric energy–momentum tensor again serves as the most appropriate local description of energy for the field [13]. The best and most recent version of field theory of gravitation given in [13] satisfies the natural requirement that all viable theories of gravity should dynamically coincide in the weak–field approximation with the linearized General Relativity i.e. gravitation should be described by the linear field $\psi_{\mu\nu}$. Furthermore the metric stress tensor derived in [13] has a number of nice properties and according to the authors, their $T^{\mu\nu}$ is the correct energy–momentum tensor for the gravitational field. However, while the linearized Lagrange equations of their theory are gauge invariant (as being equivalent to those for $\psi_{\mu\nu}$), their energy–momentum tensor in this approximation shares the defect of all the metric stress tensors for $\psi_{\mu\nu}$, i.e. breaks the gauge symmetry. Applying a physically undeniable condition that the energy–momentum tensor should have the same gauge invariance as the field equations, we conclude that also this approach to gravity does not furnish a physically acceptable notion of gravitational energy density[6]. Of course, this does not mean that in linearized General Relativity there are no conserved tensors which are gauge invariant,

[5] A different viewpoint is based on the fact that geometrical General Relativity, including the full nonlinear Einstein field equations, can be uniquely derived as a consistent self–coupled theory arising from the free linear massless spin–2 field theory in Minkowski space [15], [16], see also [17]. In this approach an energy–momentum tensor for the gravitational field arises as a part of the field equations in the form of a Noether current.

[6] In an earlier work [20] it is shown that the field–theoretical formulation of General Relativity in a Minkowski background does not provide a gauge invariant energy–momentum tensor, but the authors do not regard this fact as a defect of their approach.

symmetric and quadratic in (second) derivatives of fields, which to some extent may play the role of energy density. The famous Bel–Robinson superenergy tensor [21] is the best example.

To avoid any misunderstanding, we emphasize that we do not regard the field–theoretical approach to gravity as an alternative theory of gravitation which might in principle replace General Relativity as an adequate description of reality. If instead one views this approach as a different theory of gravity then one may claim that the "gauge" transformation actually maps one solution of field equations to another *physically distinct* solution. Then the energy density need not be gauge invariant and measurements of energy may be used to discriminate between two physically different solutions related by the "gauge" transformation (which should then be rather called a "symmetry transformation"). The gravitational field of [13] or the spin–2 field with the Wentzel Lagrangian would then be measurable quantities rather than gauge potentials. If the transformation of these fields is not an internal gauge but corresponds to a change of physical state, this raises a difficult problem of finding out a physical interpretation of it. Clearly it is not a transformation between reference frames of any kind.

Here we adopt the opposite view that the field theory approach to gravity is merely an auxiliary procedure for constructing notions which are hard to define in the framework of General Relativity. It is commonly accepted that in the weak field limit of General Relativity the spacetime metric is measurable only in a very restricted sense: if two almost Cartesian coordinate systems are related by an infinitesimal translation $x'^{\mu} = x^{\mu} + \xi^{\mu}$, then no experiment can tell the difference of their metrics while the curvature tensor has the same components in both systems. This implies that all different coordinate systems connected by this transformation actually represent the same physical reference frame and from the physical viewpoint the transformation is an internal gauge symmetry [22]. Thus, showing a mathematical equivalence of the corresponding field equations is insufficient to achieve compatibility of an approach to gravity with the linearized General Relativity. The weak field gravity should be described by a gauge potential. In consequence, any gravitational energy density should be a gauge invariant quantity.

There are numerous no–go theorems in physics and it is not unlikely that this one will be somehow circumvented as most of them have been and an acceptable notion of gravitational energy density will be ultimately defined. However, this notion cannot be expressed in terms of the metric energy–momentum tensor in a Lagrangian field theory since in the weak–field approximation the latter cannot be gauge invariant. Therefore we feel that the theorem closes one line of research of gravitational energy density. This makes the quest of this notion harder than previously.

5 Conclusions

Our approach allows one to clarify in full generality the rather surprising fact that the metric energy–momentum tensor may not possess a symmetry property of the underlying field Lagrangian. This feature, first encountered in the case of gauge symmetry of the massless linear spin-two field, turns out to be a generic one. This is due to the presence of the spacetime metric. The metric stress tensor acquires a symmetry only if the Lagrangian has this symmetry in an open neighbourhood in the space of Lorentzian metrics; if the symmetry only holds for a specific spacetime (e.g. in Minkowski space), the stress tensor cannot inherit it besides exceptional cases (when the variational derivative in the last term in eq. (21) vanishes). Whenever the symmetry transformation of the field variables depends on the metric, the stress tensor acquires the symmetry only for solutions.

As an obvious application we have used our generic method to the case of gauge fields with spins larger than one and rederived in a different way previous conclusions [2]. The case of these high spin fields has been additionally obscured by the well known fact that these theories are dynamically inconsistent in the presence of gravitation and this is closely connected with the gauge invariance breaking in a curved spacetime. We believe that our approach sheds new light on the confusing issue of these fields.

We expect that the generic picture of symmetry breaking for the metric energy–momentum tensor in Lagrangian field theories presented in this work will find further physical applications besides the case of the gauge invariance of high spin linear fields. However, even in this case our method employed to the massless spin–2 field allows to show in a simple and direct way that the field–theory approach to gravitational energy density is actually hopeless.

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References

[1] C. Aragone, S. Deser, Nuovo Cim. **3A** (1971) 709.

[2] S. Deser, J. G. McCarthy, Class. Quantum Grav. **7** (1990) L119.

[3] P.R. Anderson, Phys. Rev. **D55** (1997) 3440 [gr-qc/9609023].

[4] L.R.W. Abramo, R.H. Brandenberger, V.F. Mukhanov, Phys.Rev. **D56** (1997) 3248 [gr-qc/9704037].

[5] J.M. Pons, D.C. Salisbury, L.C. Shepley, Phys. Rev. **D55**(1997) 658 [gr-qc/9612037]; J.Math.Phys. **41** (2000) 5557 [gr-qc/9912086].

[6] A.D. Popova, A.N. Petrov, Intern. Journ. of Mod. Phys. A**3**(1988) 2651.

[7] C. Fronsdal, Phys. Rev. **D18** (1978) 3624; B. de Wit, D. Z. Freedman, Phys. Rev. **D21** (1980) 358.

[8] M. Fierz, W. Pauli, Proc. Roy. Soc. (London) **A173** (1939) 211.

[9] C. Aragone, S. Deser, Nuovo Cim. **57B** (1980) 33.

[10] G. Wentzel, *Einführung in die Quantentheorie der Wellenfelder*, F. Deuticke (Wien 1943), par. 22.

[11] R. Hakim, *An Introduction to Relativistic Gravitation*, Cambridge UP (Cambridge, 1999); H. Ohanian, R. Ruffini, *Gravitation and Spacetime*, Norton (New York, 1994).

[12] S. Deser (2001), private communication.

[13] S.V. Babak, L.P. Grishchuk, Phys. Rev. **D61** (2000), 024038 [gr-qc 9907027].

[14] J.D. Brown, S.R. Lau, J.W. York, *Action and Energy of the Gravitational Field*, preprint gr-qc/0010024.

[15] S. Deser, Gen. Rel. Grav. **1** (1970) 9.

[16] D. Boulware, S. Deser, Ann. Phys. NY **89** (1975) 193.

[17] S. Deser, Class. Quantum Grav. **4** (1987) L99.

[18] Dongsu Bak, D. Cangemi and R. Jackiw, Phys. Rev. **D49** (1994) 5173.

[19] Y. Baryshev, *Field theory of gravitation: desire and reality*, preprint gr-qc/9912003.

[20] L.P. Grishchuk, A.N. Petrov, A.D. Popova, Comm. Math. Phys. **94** (1984) 379.

[21] L. Bel, Compt. Rend. Acad. Sci. Paris **247** (1958) 1094; I. Robinson, unpublished (1958); Class. Quant. Grav. **14** (1997) 4331.

[22] C.W. Misner, K.S. Thorne, J.A. Wheeler, *Gravitation*, W. Freeman, (San Francisco 1973), chap. 18; R.M. Wald, *General Relativity*, Univ. of Chicago Press, (Chicago 1984), chap. 4.